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COMMENT

Classification of square-lattice cellular automata with respect to total magnetization

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Abstract. We check by computer simulations whether the magnetization asymptotically approaches a fixed point or a limit cycle of period two. This criterion is applied to all 65 536 nearest-neighbour automata.

The Wolfram-type [1] classification of one-dimensional cellular automata has been applied to a variety of two- and three-dimensional lattices [2]. With four and more neighbours on the square, triangular and simple cubic lattices, most of the rules do not show fixed points or limit cycles of period two. In this work [1, 2], configurations were compared with each other, and a fixed point or oscillation was established if the configuration at time $t+1$ agreed with the one at time t or $t-1$, respectively. Recently, a mean-field theory for cellular automata was developed [3] and applied to the square lattice, with the result that all rules lead either to a fixed point or to an oscillation with period two, in striking contrast to the full simulations [2]. In this mean-field theory one looks only at the magnetization or the total number of up spins. Thus in the mean-field approach a fixed point is reached if a pair of spins is always antiparallel but oscillates in its orientation. (All this work [1–3] deals with binary variables only, i.e. each lattice carries one spin which is either up or down, like in a spin-$\frac{1}{2}$ Ising model.)

The present work therefore deals with a method intermediate between the mean-field approach [3] and the full comparison of the whole configuration [2]. We still study the full set of 65 536 cellular automata correctly, i.e. without a mean-field approximation. But in the analysis we only compare the total magnetization

$$M = \sum_i S_i \qquad S_i = \pm 1$$

and no longer the whole configuration. Thus a constant magnetization now corresponds to a fixed point in this classification; and if $M(t+1) = M(t-1)$ we have a limit cycle of period two. We start with a random spin configuration, half up and half down, $L = 64$, 192, 320 and 576.

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The computer analysis, using the simulation technique of [2] for the square lattice, depends on the time T over which we require the magnetization to be constant, and it depends on the fluctuation F by which we allow consecutive magnetizations to differ if we call them constant. We check if the magnetization M agrees with that two steps before, and if the magnetization one step before agrees with that three steps ago. If at least one of them does not agree, a counter is set equal to zero. If instead $M(t) = M(t-2)$ and $M(t-1) = M(t-3)$, then the counter is increased by one. If that counter equals the parameter T the simulation stops. Then, if also $M(t) = M(t-1)$, we have a fixed point, otherwise only an oscillation of period two. If a simulation over at most t_m time steps gives for the first time a magnetization constant or oscillating over T time steps, it stops with a positive result even if simulations over a longer time might have given a change later. We take $F = DL^2$ in an $L * L$ lattice, and $D = 0$ means that we require exact agreement. Obviously, the larger D and the smaller T is, the more likely will we find a fixed point or limit cycle. Increasing the time t_m over which we continue the simulation also increases the likelihood of classifying a rule.

Quantitatively, for $T = 1$ we found that nearly all rules could be classified in this way, similar to the mean-field result [3]. However, oscillations were found to dominate in our case, whereas fixed points dominated in the mean-field approach [3]. For example, with $D = 0$, $L = 192$, $t_m = 10^4$ we found, of the 4856 different rules (taking into account various symmetries [2]), 716 to lead to a fixed point and 3855 to lead to an oscillation of period 2, whereas only 285 rules were not yet classified; when t_m was only 10^3, the undecided rules numbered 373.

The more interesting case is the opposite limit $T \to \infty$. For $D = 0$ (i.e. for exact agreement), even at $T = 2$ most of the rules are no longer classified; for $L = 192$ we found 1075 fixed points (more than before) but only 459 oscillations for $D = 0$, $t_m = 10^3$, $T = 100$ (for $T = 10$ the numbers were 1036 and 530). All others had longer oscillations.

This trend is reversed again if we allow for positive D, i.e. if we allow a 'constant' magnetization to fluctuate by an amount DL^2. In particular, we allow for random fluctuations with $D = 1/L$. Then most of the rules could be classified, with the number of non-classified rules decreasing roughly logarithmically with increasing t_m, see figure 1. After $t_m = 10^{4.5}$ iterations, 1206 rules were not classified, 3041 led to a fixed point and 609 only to oscillations for $L = 192$. These are results for one random number sequence; sample to sample fluctuations already seem quite small for $L = 192$. Increasing the system size at fixed times increases the number of unclassified rules at the expense of fixed points and oscillations; if instead we keep t_m/L fixed, the size effects are reduced appreciably.

As pointed out in [3], the square lattice cellular automata without memory studied here separate into two checkerboard sublattices, with the one sublattice influencing only the other sublattice and not itself at the next time step. We thus also looked at the magnetization of a sublattice only (one sublattice at even and the other one at odd times). Figure 1 compares our sublattice results with the previous ones for the whole sublattice, using the same random number and the same $L = 64$, allowing for fluctuations of 1.56% ($D = 1/L$). The sublattice criterion drastically reduces both the computer time and the number of unclassified rules (down to 65 for our longest time).

(Keeping $t_m/T = 10$ fixed at $L = 64$ leads to rapid convergence for $D = 0$ and $t_m \to \infty$ without indications of logarithmic effects. For $D = 1/L$, on the other hand, we get slow convergence for the same parameters, with the number of unclassified rules increasing with increasing times.)

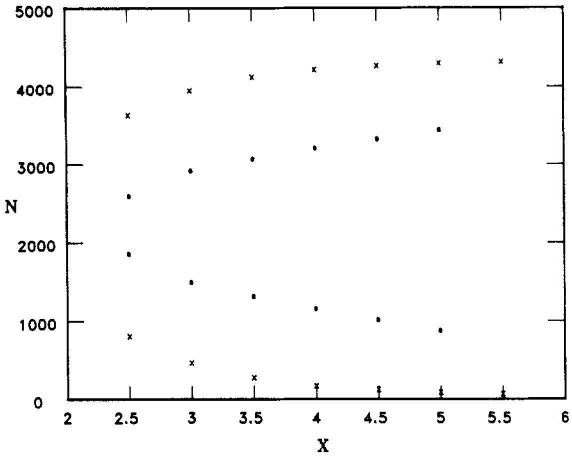

Figure 1. Variation with observation time t_m of the number N of fixed points (upper points) and of unclassified rules (lower points); $X = \log_{10}(t_m)$, $L = 64$. Dots refer to comparison of the whole lattice, crosses to comparison of a sublattice only. 4856 rules were analysed.

In summary, the classification according to the total magnetization decreases sharply the number of non-classified rules, compared with the classification [2] according to the full configuration. However, the distribution of classified behaviour into fixed points and oscillations now depends strongly on the free parameters of the simulation.

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